ME 4555 - Lecture 16 - Inverse Laplace transform

We will develop general procedure for computing the inverse Laplace transform of any transfer function (rational).

Basic overview:

1. rational transfer function: $$G(s) = \frac{3s^3 + 3s^2 + 5s - 7}{s^4 + s^3 + s^2 - 9s - 10}$$

2. Partial Fraction Expansion (PFE) of transfer function: $$G(s) = \frac{1}{s-2} + \frac{1}{s+1} + \frac{s+1}{s^2+2s+5}$$

3. Inverse Laplace transform of each term separately: $$g(t) = e^{2t} + e^{-t} + e^{-t}\cos(2t)$$

We will look into these two steps in greater detail over the next two lessons.

A fundamental Laplace transform: the exponential.

Consider the function e^{zt} where z is a constant.

$$\mathcal{L}\{e^{zt}\} = \int_0^\infty e^{-st} e^{zt}\, dt$$

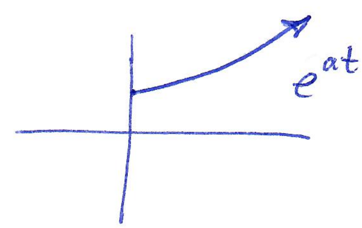
$$= \int_0^\infty e^{(z-s)t}\, dt$$

$$= \left[\frac{1}{z-s} e^{(z-s)t} \right]_{t=0}^{\infty}$$

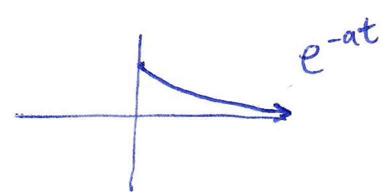
$$= \boxed{\frac{1}{s-z}}$$

If z is a real number:

$z = a > 0$: $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$



$z = -a < 0$: $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$



This is a "first order" system because the denominator has degree 1. The sign of a determines behavior.

If $z = a + ib$ (complex number), then:

$$\frac{1}{s-z} = \frac{1}{s-a-ib} = \frac{1}{(s-a)-ib} = \frac{(s-a)+ib}{[(s-a)-ib][(s-a)+ib]}$$

$$= \frac{(s-a)+ib}{(s-a)^2 + b^2} = \left[\frac{(s-a)}{(s-a)^2+b^2}\right] + i\left[\frac{b}{(s-a)^2+b^2}\right]$$

but $e^{zt} = e^{(a+ib)t} \underset{\uparrow}{=} e^{at}(\cos bt + i \sin bt)$

Euler's formula

Therefore:

$$\mathcal{L}\left\{e^{at}\cos bt + i e^{at}\sin bt\right\} = \left[\frac{(s-a)}{(s-a)^2+b^2}\right] + i\left[\frac{b}{(s-a)^2+b^2}\right]$$

Equating real and imaginary parts, we find:

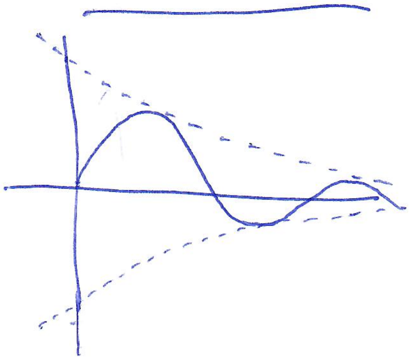
$$\mathcal{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

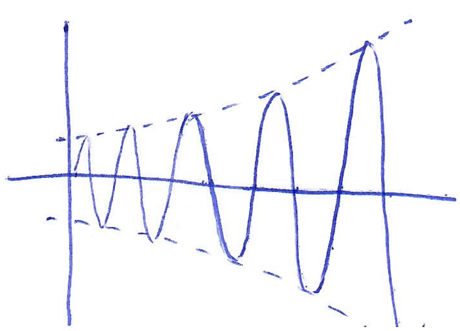
★ sign of a determines whether function grows or decays exponentially

★ magnitude of b determines frequency of oscillation.

$a<0$, b small



$a>0$, b large



Every polynomial with real coefficients has:

- n roots, where n is the degree of the polynomial.
- if $a+ib$ is a root, so is $a-ib$ (roots occur in complex conjugate pairs).
- if roots are $z_1, z_2, \ldots, z_n$ then we can write the polynomial as:

$$p(x) = \underset{\substack{\uparrow \\ \text{real constant.}}}{k}(x-z_1)(x-z_2)\cdots(x-z_n)$$

This is known as the "fundamental theorem of algebra".

For roots that occur as conjugate pairs, the factors look like:

$$\begin{aligned}
&(x-(a+ib))(x-(a-ib)) \\
&= ((x-a)+ib)((x-a)-ib) \\
&= (x-a)^2-(ib)^2 \\
&= (x-a)^2+b^2
\end{aligned}$$

So if the real roots are $\{r_1, \ldots, r_p\}$ and complex roots are $\{a_1 \pm ib_1, \ldots a_q \pm ib_q\}$, the factorization looks like:

$$p(x) = k(x-r_1)\cdots(x-r_p)\left((x-a_1)^2+b_1^2\right)\cdots\left((x-a_q)^2+b_q^2\right)$$

Partial fraction expansion

$$\frac{B(x)}{A(x)} = \, ?$$

1st step: Make sure $\deg(B) < \deg(A)$. Otherwise, perform polynomial division.

Example: $\dfrac{x^3 + x - 1}{x^2 + 3x + 2}$ $\dfrac{(\deg 3)}{(\deg 2)}$ 3 is larger than 2 so divide!

$$\begin{array}{r l}
 & x - 3 \longrightarrow \text{quotient.} \\
x^2 + 3x + 2 \, \big) & x^3 + 0x^2 + x - 1 \\
- & x^3 + 3x^2 + 2x \\
\hline
 & -3x^2 - x - 1 \\
- & -3x^2 - 9x - 6 \\
\hline
 & 8x + 5 \longrightarrow \text{remainder.}
\end{array}$$

Therefore $$\frac{x^3 + x - 1}{x^2 + 3x + 2} = x - 3 + \underbrace{\frac{8x + 5}{x^2 + 3x + 2}}$$

now we will perform expansion on this term; note denominator has larger degree than numerator!

2nd step: Factor the denominator: (no need to factor numerator)

$$\frac{8x+5}{x^2+3x+2} = \frac{8x+5}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$$

PFE will have this form.

Note: this is what happens when the poles (roots of the denominator polynomial) are <u>real</u> and <u>distinct</u>. We will treat the complex case later.

Now, solve for a and b so that the equation holds:

$$\frac{8x+5}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2} = \frac{a(x+2)+b(x+1)}{(x+1)(x+2)}.$$

So we need to pick a, b such that $a+b=8$ and $2a+b=5$.

$$= \frac{(a+b)x+(2a+b)}{(x+1)(x+2)}$$

$$\begin{aligned} a+b &= 8 \\ 2a+b &= 5 \end{aligned} \implies a=-3,\ b=11.$$

therefore:

$$\boxed{\frac{x^3+x-1}{x^2+3x+2} = x-3 - \frac{3}{x+1} + \frac{11}{x+2}}$$